Terminal region enlargement of a stabilizing NMPC design for a multicopter

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Outline

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Introduction – NMPC with terminal ingredients

NMPC problem formulation[1]

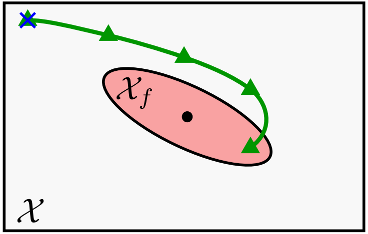
$$\bar{\mathbf{u}}_t^*(\cdot) = \arg\min_{\bar{\mathbf{u}}_t(\cdot)} \int_t^{t+T_p} \ell(\bar{\mathbf{x}}_t(s), \bar{\mathbf{u}}_t(s))ds + F(\bar{\mathbf{x}}_t(t+T_p)),$$

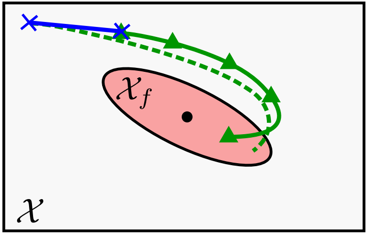
subject to:
$$\dot{\bar{\mathbf{x}}}_t = f(\bar{\mathbf{x}}_t, \bar{\mathbf{u}}_t),$$
$$\bar{\mathbf{u}}_t(s) \in \mathcal{U},\ s \in [t, t+T_p],$$
$$\bar{\mathbf{x}}_t(t) = \mathbf{x}(t),$$
$$\bar{\mathbf{x}}_t(t+T_p) \in \mathcal{X}_f.$$

$\mathcal{U}$: input constraint set

$\mathcal{X}_f$: terminal state constraint set

$\mathcal{X}$: state constraint set





[1] Grüne and Pannek 2017; Chen and Allgöwer 1998

Introduction – Stability conditions

Four conditions that ensure the asymptotic stability[2]

1. $\mathcal{X}_f$ closed, $\mathbf{0} \in \mathcal{X}_f$, and $\mathcal{X}_f \subset \mathcal{X}$
2. $\forall \mathbf{x} \in \mathcal{X}_f,\ \mathbf{u}_{\text{loc}}(\mathbf{x}) \in \mathcal{U}$
3. $\forall \mathbf{x} \in \mathcal{X}_f,\ f(\mathbf{x}, \mathbf{u}_{\text{loc}}(\mathbf{x})) \in \mathcal{X}_f$
4. $\forall \mathbf{x} \in \mathcal{X}_f,\ \overset{*}{F}(\mathbf{x}, \kappa_f(\mathbf{x})) + \ell(\mathbf{x}, \mathbf{u}_{\text{loc}}(\mathbf{x})) \leq 0$
   - $[F(\mathbf{x}(k), \mathbf{u}_{\text{loc}}(\mathbf{x})) - F(\mathbf{x}(k))] + \ell(\mathbf{x}(k), \mathbf{u}_{\text{loc}}(\mathbf{x}(k))) \leq 0$ (discrete)
   - $\frac{dF}{dt}(f(\mathbf{x}(t)) + \ell(\mathbf{x}(t), \mathbf{u}_{\text{loc}}(\mathbf{x}(t))) \leq 0$ (continuous)

[2] David Q. Mayne et al. (2000). "Constrained Model Predictive Control: Stability and Optimality". In: *Automatica* 36.6, pp. 789–814. DOI: 10.1016/S0005-1098(99)00214-9

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Challenges

How to design such a terminal set $\mathcal{X}_f$?

- Chen and Allgöwer 1998 systemically solve an optimization problem to obtain a terminal set with no systematic way to tune the size of set.
- N. T. Nguyen, Prodan, and Lefevre 2021 employ the feedback linearization as the local controller and construct the terminal set for multicopters.
- Eyüboğlu and Lazar 2022 transform constraints into LMIs and solve them with MATLAB.
- Comelli et al. 2023 replace the terminal set with a pair of simpler inner-outer sets that are unnecessary to be invariant.

Motivation

How to (maximally) enlarge that terminal set?

- De Doná et al. 2002 use a saturated linear feedback controller as a local controller and modify the terminal ingredients based on their new local controller.
- Cannon, Kouvaritakis, and Deshmukh 2004 use the concept of partial invariant sets and solve offline linear programming problems to maximize their volumes.
- Limon, Alamo, and Camacho 2005 compute the sequence of reachable sets using the inner-approximations of one-step sets to construct a contractive terminal set.
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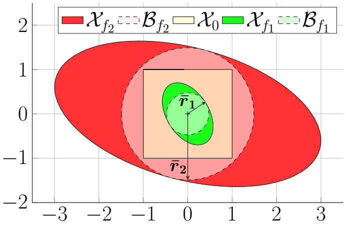
Problem ?

Relying on optimization approach $\Rightarrow$ computationally intractable.

Motivation

Semi-globally asymptotic stability

A system is semi-globally stabilizable[3] to an equilibrium point $\boldsymbol{x}_e$ by means of a class $\mathcal{F}$ of feedback control laws if, for any a priori determined compact set $\mathcal{X}_0$ of initial conditions, there exists a control law in $\mathcal{F}$ that makes $\boldsymbol{x}_e$ asymptotically stable with a domain of attraction that contains $\mathcal{X}_0$.



[3] J H Braslavsky and R H Middleton (1996). "Global and Semi-Global Stabilizability in Certain Cascade Nonlinear Systems". In: *IEEE Transactions on Automatic Control* 41.6, p. 6. DOI: 10.1109/9.506242

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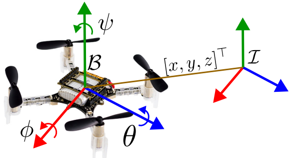
Multicopter dynamical model

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \underbrace{\begin{bmatrix}0\\0\\-g\end{bmatrix} + \begin{bmatrix}c(\phi)s(\theta)c(\psi)+s(\phi)s(\psi)\\ c(\phi)s(\theta)s(\psi)-s(\phi)c(\psi)\\ c(\phi)c(\theta)\end{bmatrix}T}_{h(\mathbf{u},\psi)}$$

- $\boldsymbol{\xi} \triangleq [x, y, z]^\top$: 3D position
- g: gravity
- (ϕ, θ, ψ): Euler angles
- $T \in \mathbb{R}_+$: normalized input thrust



$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

- $\mathbf{x} \triangleq \left[\boldsymbol{\xi}^\top, \dot{\boldsymbol{\xi}}^\top\right]^\top \in \mathbb{R}^6$: state
- $\mathbf{u} \triangleq [T, \phi, \theta]^\top \in \mathbb{R}^3$: input
- $f(\cdot) \triangleq [\dot{\boldsymbol{\xi}}^\top, h^\top(\mathbf{u}, \psi)]^\top$

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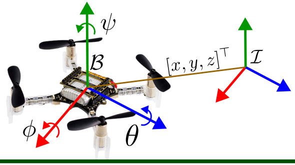

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$$\mathbf{u}(t) \in \mathcal{U} = \left\{(T, \phi, \theta) : 0 \le T \le T_{\max}, |\phi|, |\theta| \le \epsilon_{\max}\right\},$$

$$\mathbf{x}_e = \mathbf{0}, \ \mathbf{u}_e = [g, 0, 0]^\top.$$

Feedback linearization[4]

$$\mathbf{u}_{\mathsf{FL}}(\mu_\xi, \psi) \triangleq [T_{\mathsf{FL}}(\mu_\xi), \phi_{\mathsf{FL}}(\mu_\xi, \psi), \theta_{\mathsf{FL}}(\mu_\xi, \psi)]^\top$$

- ψ: yaw angle
- $\mu_\xi \triangleq [\mu_x, \mu_y, \mu_z]^\top$: virtual control input

$$T_{\mathsf{FL}}(\mu_\xi) = \sqrt{\mu_x^2 + \mu_y^2 + (\mu_z + g)^2}$$

$$\phi_{\mathsf{FL}}(\mu_\xi; \psi) = \arcsin\left(\frac{\mu_x s(\psi) - \mu_y c(\psi)}{\sqrt{\mu_x^2 + \mu_y^2 + (\mu_z + g)^2}}\right)$$

$$\theta_{\mathsf{FL}}(\mu_\xi; \psi) = \arctan\left(\frac{\mu_x c(\psi) + \mu_y s(\psi)}{\mu_z + g}\right)$$

[4] Ngoc Thinh Nguyen, Ionela Prodan, and Laurent Lefèvre (June 2020). "Flat Trajectory Design and Tracking with Saturation Guarantees: A Nano-Drone Application". In: *International Journal of Control* 93.6, pp. 1266–1279. DOI: 10.1080/00207179.2018.1502474

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If $\mu_z \geq -g$:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mu_\xi,$$

with $A = \begin{bmatrix} 0_{3\times3} & \boldsymbol{I}_3 \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}$ and $B = [0_{3\times3}, \boldsymbol{I}_3]^\top$

Input constraint admissible set

$$\mathcal{X}_{\text{FL}} = \left\{ |\mu_x| \leq U_x,\ |\mu_y| \leq U_y,\ |\mu_z| \leq U_z \right\}$$

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Feedback linearization as a local controller in NMPC

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$$\mu_\xi = \mathbf{x}_e + K\mathbf{x} \overset{\mathbf{x}_e=\mathbf{0}}{=} K\mathbf{x} \Rightarrow \mathbf{u}_{\mathsf{loc}}(\mathbf{x}) \triangleq \mathbf{u}_{\mathsf{FL}}(K\mathbf{x}, \psi)$$

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$$K = \begin{bmatrix} K_{p_x} & 0 & 0 & K_{d_x} & 0 & 0 \\ 0 & K_{p_y} & 0 & 0 & K_{d_y} & 0 \\ 0 & 0 & K_{p_z} & 0 & 0 & K_{d_z} \end{bmatrix}$$

$$\dot{\mathbf{x}} = (A+BK)\mathbf{x} = A_K\mathbf{x}$$

Problem set up – NMPC with terminal ingredients

Stage cost

$$\ell(\mathbf{x},\mathbf{u}) \triangleq \|\mathbf{x}-\mathbf{x}_e\|_Q^2 + \|\mathbf{u}-\mathbf{u}_e\|_R^2, \quad Q \in \mathbb{S}_{++}^6, R \in \mathbb{S}_+^3\text{: to be defined}$$

Terminal cost

$$F(\mathbf{x}) \triangleq \|\mathbf{x}-\mathbf{x}_e\|_P^2, \quad P \in \mathbb{S}_{++}^6\text{: obtained by solving } A_K^\top P + PA_K + M = \mathbf{0}$$

Terminal set[5]

$$\mathcal{X}_f = \{\mathbf{x} \in \mathbb{R}^6 : \mathbf{x}^\top P\mathbf{x} \leq \delta\}, \text{ with } \delta = \lambda_{\min}(P)r^2, \; r^2 = \min_{q\in\{x,y,z\}} \left\{ \frac{U_q^2}{K_{p_q}^2 + K_{d_q}^2} \right\}$$

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NMPC with terminal ingredients

Boundness of the FL local controller [6]

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$$\|\mathbf{u}_{\text{loc}}(\mathbf{x}) - \mathbf{u}_e\|^2 \leq \mathbf{x}^\top (K^\top K + 2\Gamma)\mathbf{x}$$

$$\Gamma = \frac{1}{(-U_z + g)^2} K_{xy}^\top K_{xy}$$

$$K_{xy} = \begin{bmatrix} K_{p_x} & 0 & 0 & K_{d_x} & 0 & 0 \\ 0 & K_{p_y} & 0 & 0 & K_{d_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\|\mathbf{u}_{\text{loc}}(\mathbf{x}) - \mathbf{u}_e\|^2 \leq \mathbf{x}^\top (K^\top K + 2\Gamma)\mathbf{x}$$

$$\Gamma = \frac{1}{(-U_z + g)^2} K_{xy}^\top K_{xy}$$

$$K_{xy} = \begin{bmatrix} K_{p_x} & 0 & 0 & K_{d_x} & 0 & 0 \\ 0 & K_{p_y} & 0 & 0 & K_{d_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[6] Ngoc Thinh Nguyen, Ionela Prodan, and Laurent Lefevre (2021). "Stability Guarantees for Translational Thrust-Propelled Vehicles Dynamics Through NMPC Designs". In: *IEEE Transactions on Control Systems Technology* 29, pp. 207–219. DOI: 10.1109/TCST.2020.2974146

NMPC with terminal ingredients

$$\frac{d}{dt}F(f(\mathbf{x})) + \ell(\mathbf{x}, \mathbf{u}_{\text{loc}}(\mathbf{x})) = (\dot{\mathbf{x}}^\top P\mathbf{x} + \mathbf{x}^\top P\dot{\mathbf{x}}) + \mathbf{x}^\top Q\mathbf{x} + (\mathbf{u}_{\text{loc}}(\mathbf{x}) - \mathbf{u}_e)^\top R(\mathbf{u}_{\text{loc}}(\mathbf{x}) - \mathbf{u}_e)$$

$$\leq \mathbf{x}^\top (A_K^\top P + PA_K)\mathbf{x} + \mathbf{x}^\top \underbrace{[Q + \lambda_{\max}(R)(K^\top K + 2\Gamma)]}\mathbf{x}$$

$$= \mathbf{x}^\top \underbrace{(A_K^\top P + PA_K)}\mathbf{x} + \qquad \mathbf{x}^\top Q^* \mathbf{x}$$

$$= \mathbf{x}^\top (\quad -M + Q^*)\mathbf{x} \leq 0$$

$$\ell(\mathbf{x}, \mathbf{u}) \triangleq \|\mathbf{x} - \mathbf{x}_e\|_Q^2 + \|\mathbf{u} - \mathbf{u}_e\|_R^2, \quad F(\mathbf{x}) \triangleq \|\mathbf{x} - \mathbf{x}_e\|_P^2$$

$$\|\mathbf{u}_{\text{loc}}(\mathbf{x}) - \mathbf{u}_e\|^2 \leq \mathbf{x}^\top (K^\top K + 2\Gamma)\mathbf{x}$$

$$\Rightarrow \|\mathbf{u}_{\text{loc}}(\mathbf{x}) - \mathbf{u}_e\|_R^2 \leq \lambda_{\max}(R)\|\mathbf{u}_{\text{loc}}(\mathbf{x}) - \mathbf{u}_e\|^2 \leq \mathbf{x}^\top [\lambda_{\max}(R)(K^\top K + 2\Gamma)]\mathbf{x}$$

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Explicit solution of the Lyapunov equation

Lemma (how to find M)

$\exists$ a square diagonal matrix $M \in \mathbb{S}^6_{++}$,

$$M = \text{diag}\{m_x, m_y, m_z, m_{v_x}, m_{v_y}, m_{v_z}\},$$

and a symmetric matrix $Q^* \in \mathbb{S}^6_{++}$,

$$Q^* = \begin{bmatrix} \text{diag}\{Q^*_{1_x}, Q^*_{1_y}, Q^*_{1_z}\} & \text{diag}\{Q^*_{3_x}, Q^*_{3_y}, Q^*_{3_z}\} \\ \text{diag}\{Q^*_{3_x}, Q^*_{3_y}, Q^*_{3_z}\} & \text{diag}\{Q^*_{2_x}, Q^*_{2_y}, Q^*_{2_z}\} \end{bmatrix},$$

where

$$Q^* \triangleq Q + \lambda_{\max}(R)(K^\top K + 2\Gamma),$$

satisfying $M \succcurlyeq Q^* \succ \mathbf{0}$.

Explicit solution of the Lyapunov equation

Sketch of the proof

- $\mathrm{spec}(K^\top K) = \{0, 0, 0, K_{p_x}^2 + K_{d_x}^2, K_{p_y}^2 + K_{d_y}^2, K_{p_z}^2 + K_{d_z}^2\} \Rightarrow K^\top K \succcurlyeq \mathbf{0}$
- $\mathrm{spec}(K_{xy}^\top K_{xy}) = \{0, 0, 0, 0, K_{p_x}^2 + K_{d_x}^2, K_{p_y}^2 + K_{d_y}^2\} \Rightarrow K_{xy}^\top K_{xy} \succcurlyeq \mathbf{0} \Rightarrow \Gamma \succcurlyeq \mathbf{0}$
- $R \in \mathbb{S}_+^3 \Rightarrow R \succcurlyeq \mathbf{0} \Rightarrow \lambda_{\max}(R) \geq 0$

$\Rightarrow \lambda_{\max}(R)(K^\top K + 2\Gamma) \succcurlyeq \mathbf{0}$
$\Rightarrow Q^* \triangleq Q + \lambda_{\max}(R)(K^\top K + 2\Gamma) \succ \mathbf{0}$ (since $Q \in \mathbb{S}_{++}^6 \succ \mathbf{0}$).

Sketch of the proof (cont.)

Now, we want: $M \succcurlyeq Q^* \Leftrightarrow (M - Q^*) \succcurlyeq \mathbf{0}$

We choose: $m_q \geq Q_{1_q}^* + |Q_{3_q}^*| > 0,\ m_{v_q} \geq Q_{2_q}^* + |Q_{3_q}^*| > 0$

$\Rightarrow M \succcurlyeq Q^* \succ \mathbf{0}.$

Explicit solution of the Lyapunov equation

Sketch of the proof

- $\mathrm{spec}(K^\top K) = \{0, 0, 0, K_{p_x}^2 + K_{d_x}^2, K_{p_y}^2 + K_{d_y}^2, K_{p_z}^2 + K_{d_z}^2\} \Rightarrow K^\top K \succcurlyeq \mathbf{0}$
- $\mathrm{spec}(K_{xy}^\top K_{xy}) = \{0, 0, 0, 0, K_{p_x}^2 + K_{d_x}^2, K_{p_y}^2 + K_{d_y}^2\} \Rightarrow K_{xy}^\top K_{xy} \succcurlyeq \mathbf{0} \Rightarrow \Gamma \succcurlyeq \mathbf{0}$
- $R \in \mathbb{S}_+^3 \Rightarrow R \succcurlyeq \mathbf{0} \Rightarrow \lambda_{\mathsf{max}}(R) \geq 0$

$\Rightarrow \lambda_{\mathsf{max}}(R)(K^\top K + 2\Gamma) \succcurlyeq \mathbf{0}$
$\Rightarrow Q^* \triangleq Q + \lambda_{\mathsf{max}}(R)(K^\top K + 2\Gamma) \succ \mathbf{0}$ (since $Q \in \mathbb{S}_{++}^6 \succ \mathbf{0}$).

Sketch of the proof (cont.)

Now, we want: $M \succcurlyeq Q^* \Leftrightarrow (M - Q^*) \succcurlyeq \mathbf{0}$

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$$\Rightarrow M \succcurlyeq Q^* \succ \mathbf{0}.$$

Explicit solution of the Lyapunov equation

Proposition (how to find P having a defined M)

The solution of the Lyapunov equation is defined as a symmetric matrix $P \in \mathbb{S}^6_{++}$:

$$P \triangleq \begin{bmatrix} [P_1] & [P_3] \\ [P_3] & [P_2] \end{bmatrix} = \begin{bmatrix} \operatorname{diag}\{P_{1_x}, P_{1_y}, P_{1_z}\} & \operatorname{diag}\{P_{3_x}, P_{3_y}, P_{3_z}\} \\ \operatorname{diag}\{P_{3_x}, P_{3_y}, P_{3_z}\} & \operatorname{diag}\{P_{2_x}, P_{2_y}, P_{2_z}\} \end{bmatrix},$$

whose entries are given by:

$$P_{1_q} = \frac{1}{2}\left(\frac{K_{d_q}}{K_{p_q}} - \frac{1}{K_{d_q}}\right) m_q + \frac{K_{p_q}}{2K_{d_q}} m_{v_q}, \tag{1a}$$

$$P_{2_q} = \frac{m_q}{2K_{p_q}K_{d_q}} - \frac{m_{v_q}}{2K_{d_q}}, \quad P_{3_q} = -\frac{m_q}{2K_{p_q}}, \tag{1b}$$

for $q \in \{x, y, z\}$.

Explicit solution of the Lyapunov equation

Lemma (eigenvalues of P)

The spectrum of the matrix $P \in \mathbb{S}^6_{++}$ composes of six positive eigenvalues:

$$\mathrm{spec}(P) = \{\lambda_{1_q}, \lambda_{2_q} : q \in \{x, y, z\}\},$$

where each pair of eigenvalues is explicitly given by:

$$\{\lambda_{1_q}, \lambda_{2_q}\} = \left\{ \frac{1}{2} \left(P_{1_q} + P_{2_q} \pm \sqrt{(P_{1_q} - P_{2_q})^2 + 4P_{3_q}^2} \right) \right\},$$

and $\{P_{1_q}, P_{2_q}, P_{3_q} : q \in \{x, y, z\}\}$ are from (1).

Terminal region enlargement

Ellipsoid terminal set

$$\mathcal{X}_f = \{\mathbf{x} \in \mathbb{R}^6 : \mathbf{x}^\top P \mathbf{x} \leq \delta\},$$

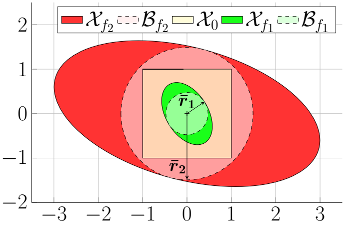
with $\delta = \lambda_{\mathsf{min}}(P) r^2$.

Terminal ball

$$\mathcal{B}_f = \left\{\mathbf{x} \in \mathbb{R}^6 \mid \|\mathbf{x}\|^2 \leq \frac{\lambda_{\mathsf{min}}(P)}{\lambda_{\mathsf{max}}(P)} r^2 \right\}.$$

Lemma

$\mathcal{B}_f \subseteq \mathcal{X}_f$.



Terminal region enlargement

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Lemma

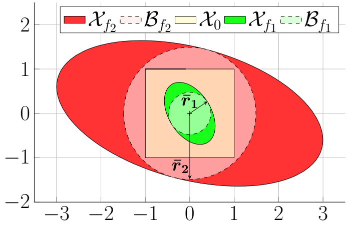
$\mathcal{B}_f \subseteq \mathcal{X}_f$.

Proof

- $\mathbf{x} \in \mathcal{B}_f \Rightarrow \lambda_{\text{max}}(P)\|\mathbf{x}\|^2 \leq \lambda_{\text{min}}(P) r^2$
- $\mathbf{x} \in \mathbb{R}^6 \Rightarrow \|\mathbf{x}\|_P^2 \leq \lambda_{\text{max}}(P)\|\mathbf{x}\|^2$

$\Rightarrow \|\mathbf{x}\|_P^2 \leq \lambda_{\text{max}}(P)\|\mathbf{x}\|^2 \leq \lambda_{\text{min}}(P) r^2$

$\Rightarrow \mathcal{B}_f \subseteq \mathcal{X}_f$



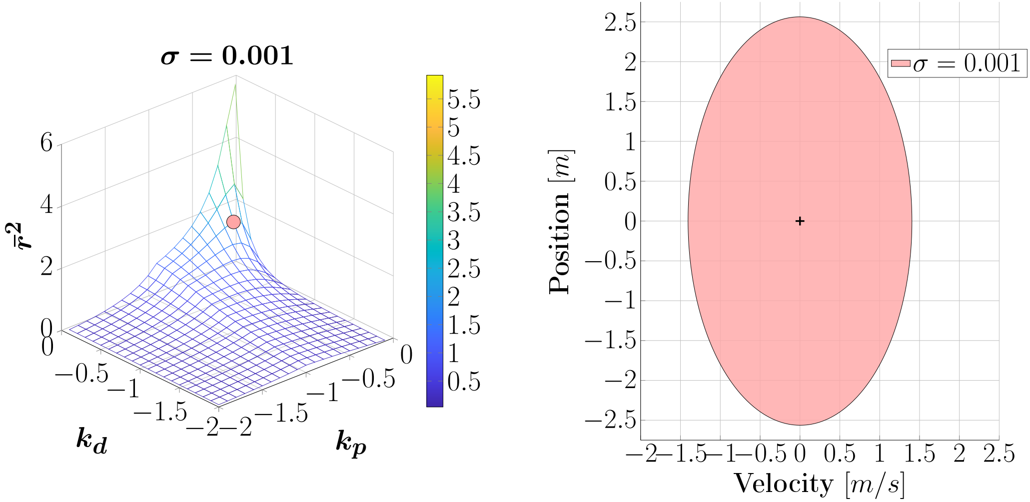
Terminal region enlargement

Assumption

$$K_{p_q} \triangleq k_p,\ K_{d_q} \triangleq k_d,\ m_q \triangleq m_1,\ m_{v_q} \triangleq m_2,\ \forall q \in \{x, y, z\}.$$

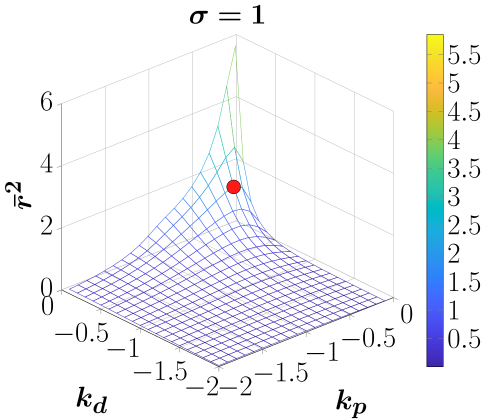
$$\Rightarrow \lambda_{\mathsf{min}}(P) = \min_{q\in\{x,y,z\}} \{\lambda_{1_q}, \lambda_{2_q}\},\ \lambda_{\mathsf{max}}(P) = \max_{q\in\{x,y,z\}} \{\lambda_{1_q}, \lambda_{2_q}\}.$$

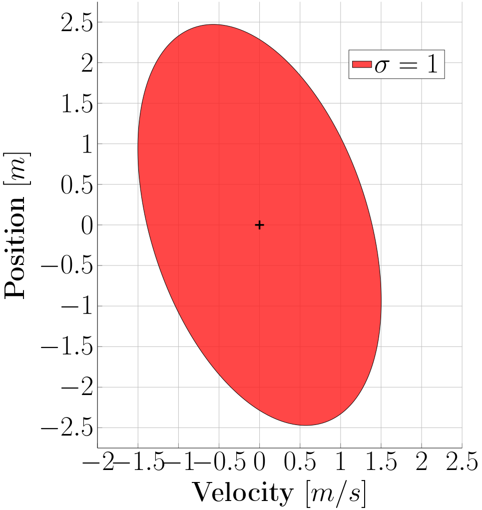
Terminal region enlargement



$m_1 = 30, m_2 = 30000, \sigma = m_1/m_2, k_p = k_d = -0.3, \bar{r}^2 = 1.9731$

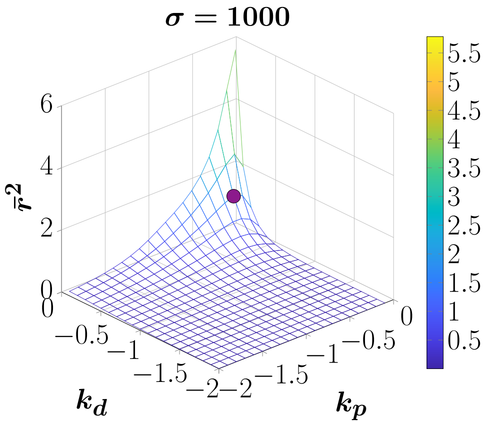
Terminal region enlargement

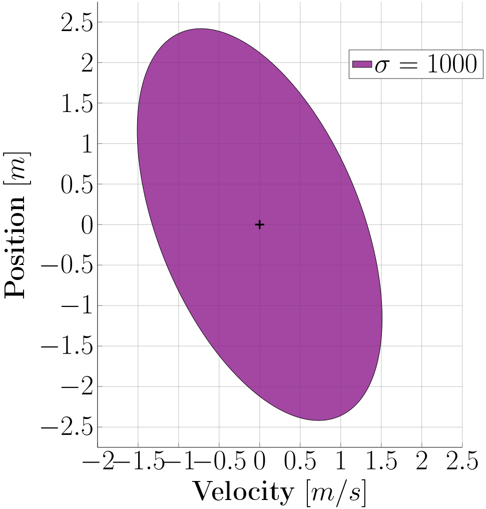




$m_1 = 30, m_2 = 30, \sigma = m_1/m_2, k_p = k_d = -0.3, \bar{r}^2 = 1.7951$

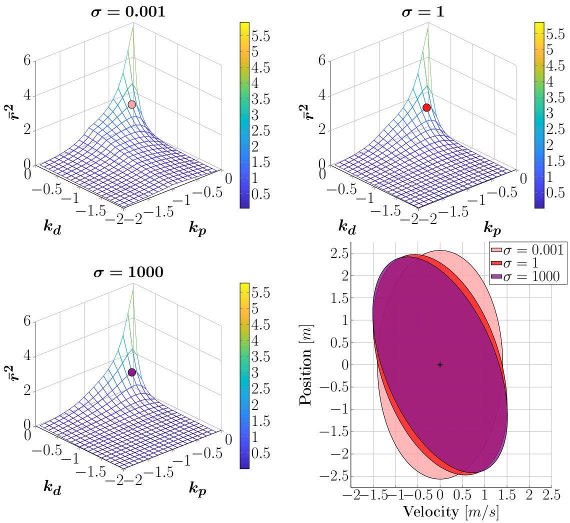
Terminal region enlargement





$$m_1 = 30000, m_2 = 30, \sigma = m_1/m_2, k_p = k_d = -0.3, \bar{r}^2 = 1.5638$$

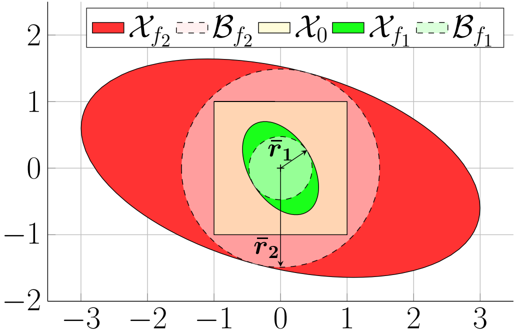
Terminal region enlargement



Terminal region enlargement

Proposition (infinity radius)

The radius of the ball $\mathcal{B}_f$ can be enlarged to infinity with appropriate feedback gains K_{p_q}, K_{d_q} and the matrix M satisfying the previous Assumption.



Terminal region enlargement

Sketch of the proof

$$\lim_{(k_p,k_d)\to(0^-,0^-)} \bar{r}^2 = \cdots = \lim_{(k_p,k_d)\to(0^-,0^-)} \frac{-\sigma k_d^2 k_p - k_p^3 + 2k_p^2\sigma - k_p\sigma^2}{\sigma^2(k_p^2+k_d^2)} U_{\mathsf{min}}^2$$

$$-\sigma k_d^2 k_p - k_p^3 + 2k_p^2\sigma > 0$$

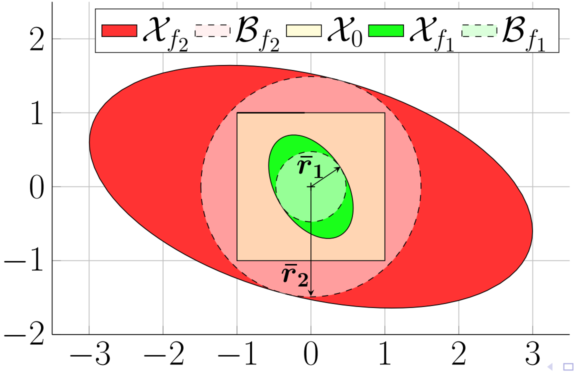
$$\lim_{(k_p,k_d)\to(0^-,0^-)} \frac{-k_p}{k_p^2+k_d^2} = +\infty$$

when $k_d^2 \to 0$ faster than $k_p \to 0$.

Terminal region enlargement

Proposition (semi-globally asymptotic stability)

Suppose that the Assumptions are satisfied, the ellipsoid $\mathcal{X}_f$ can serve as the terminal set for the NMPC problem to achieve the semi-globally asymptotic stability.



Semi-globally asymptotic stabilizing NMPC – Algorithm

1. **data** The compact set $\mathcal{X}_0$ contains the equilibrium state $\mathbf{x}_e$, U_q $(q \in \{x, y, z\})$
2. Calculate $r_0 = d(\mathbf{x}_e, \mathcal{X}_0)$
3. Choose $Q \in \mathbb{S}^6_{++}$, $R \in \mathbb{S}^3_+$
4. Construct K and calculate Γ by solving $\bar{r} \geq r_0$ for $k_p, k_d < 0$, with:
   $\bar{r} =$ compute_r_bar(k_p, k_d)
   1. Calculate $Q^* = Q + \lambda_{\text{max}}(R)(K^\top K + 2\Gamma)$
   2. Specify $M \succcurlyeq Q^*$
   3. Determine P in the terminal cost as a function of k_p, k_d, m_1, and m_2
   4. Calculate $r^2 = \min\limits_{q \in \{x,y,z\}} \left\{ \frac{U_q^2}{{k_p}^2 + k_d^2} \right\}$
   5. Calculate the radius of $\mathcal{B}_f$: $\bar{r}^2 = \frac{\lambda_{\text{min}}(P)}{\lambda_{\text{max}}(P)} r^2$
5. Construct the terminal set $\mathcal{X}_f$
6. **return** P, $\mathcal{X}_f$
7. Choose the MPC prediction horizon T_P which guarantees the recursive feasibility
8. Solve the optimization problem
9. **result** NMPC solution $\bar{\mathbf{u}}_t^*$

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Simulation scenarios

- Scenario 1: quasi-infinite MPC (qsMPC-Chen and Allgöwer 1998)
- Scenario 2: the initial state is **outside** of the terminal set[7]
- Scenario 3: the initial state is **inside** the terminal set[7]

[7] Huu Thien Nguyen, Ngoc Thinh Nguyen, and Ionela Prodan (Jan. 2024). "Notes on the Terminal Region Enlargement of a Stabilizing NMPC Design for a Multicopter". In: *Automatica* 159, p. 111375. DOI: 10.1016/j.automatica.2023.111375

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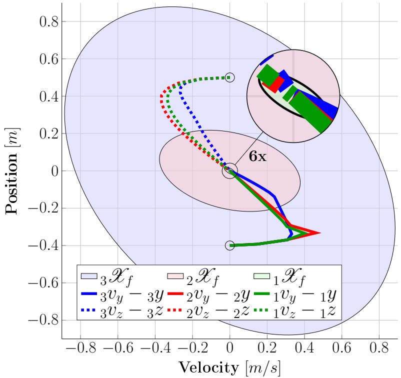
Simulation parameters

Parameters	Scenario 1 (qsMPC)	Scenario 2 (outer)	Scenario 3 (inner)
Q, R	$10\boldsymbol{I}_6, \quad \boldsymbol{I}_3$		
(k_p, k_d)	$(-2, -2)$	$(-2, -2)$	$(-0.75, -0.75)$
$\max\{Q^*_{1_q} + \lvert Q^*_{3_q}\rvert\}$	–	18.2103	11.1546
$\max\{Q^*_{2_q} + \lvert Q^*_{3_q}\rvert\}$	–	18.2103	11.1546
M	–	$M = \begin{bmatrix} 20\boldsymbol{I}_3 & \mathbf{0} \\ \mathbf{0} & 30\boldsymbol{I}_3 \end{bmatrix}$	$M = \begin{bmatrix} 20\boldsymbol{I}_3 & \mathbf{0} \\ \mathbf{0} & 30\boldsymbol{I}_3 \end{bmatrix}$
P	P_{qs}	$P = \begin{bmatrix} 30\boldsymbol{I}_3 & 5\boldsymbol{I}_3 \\ 5\boldsymbol{I}_3 & 10\boldsymbol{I}_3 \end{bmatrix}$	$P = \begin{bmatrix} 38.(3)\boldsymbol{I}_3 & 13.(3)\boldsymbol{I}_3 \\ 13.(3)\boldsymbol{I}_3 & 37.(7)\boldsymbol{I}_3 \end{bmatrix}$
r	–	$r = 0.3845$	$r = 1.0253$
$\bar{r}$	–	$\bar{r} = 0.2045$	$\bar{r} = 0.7111$
κ, α	$0.95, \quad 0.0687$	–	–
T_p	$1.9s$ (19 steps)	$1.1s$ (11 steps)	$0.2s$ (2 steps)

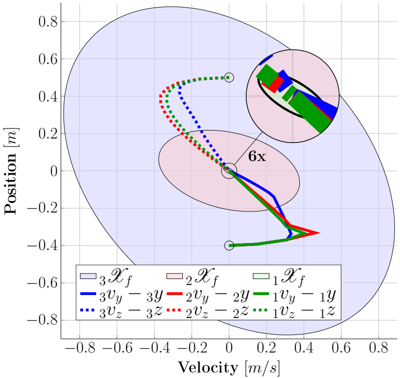
$$\mathcal{U} = \{0 \leq T \leq 2g, \ \lvert\phi\rvert \leq 10^\circ, \ \lvert\theta\rvert \leq 10^\circ\}$$



The trajectories projected onto the $y - z$ plane.

	Scen. 1	Scen. 2	Scen. 3
vol $(\mathcal{X}_f)$	4.3766×10^{-10}	0.0025	$\mathbf{2.0028}$

Simulation results



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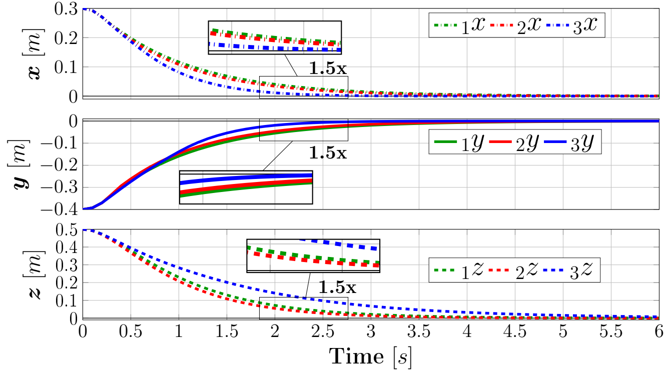
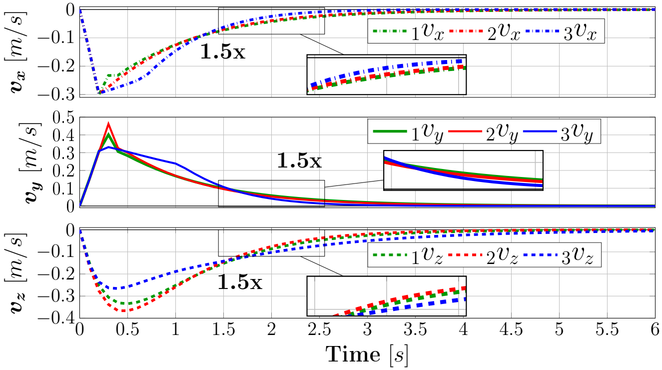


Multicopter actual motion for the 3 scenarios.

	Scen. 1	Scen. 2	Scen. 3
x	2.7	2.5	1.8
y	3	2.8	2
z	3.1	2.8	4.7

Table: The 2% settling time $(t_s\ [s])$.

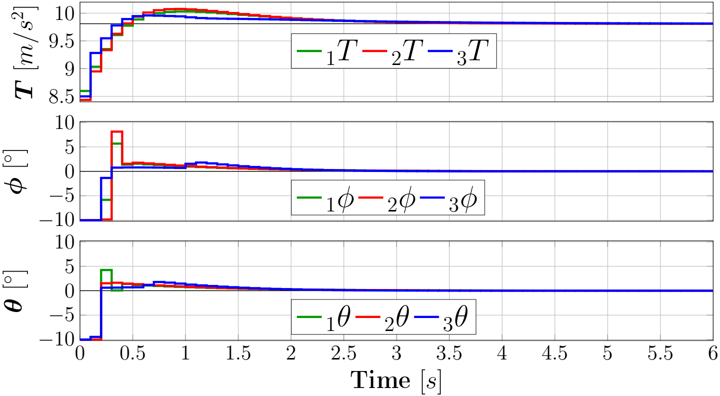


Multicopter velocity for the 3 scenarios.

	Scen. 1	Scen. 2	Scen. 3
v_x	2.8	2.6	2.1
v_y	3.1	2.9	2.4
v_z	3.3	3.0	4.3

Table: The 2% settling time (t_s $[s]$).

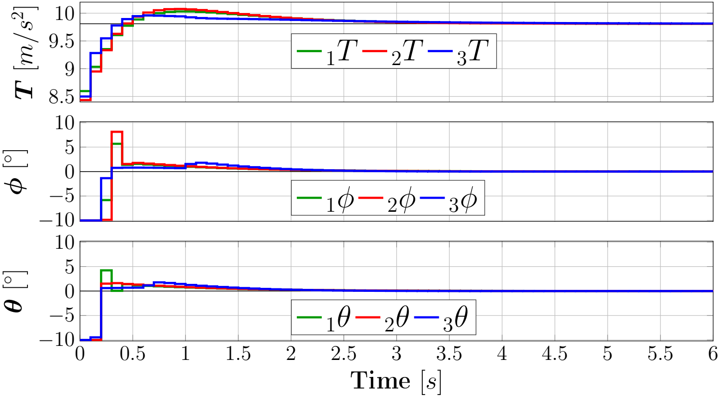
Simulation results



Multicopter control inputs for the 3 scenarios.

	Scen. 1	Scen. 2	Scen. 3
Energy $E\ [m^2/s^3]$	588.6654	588.9833	**588.3076**

Simulation results



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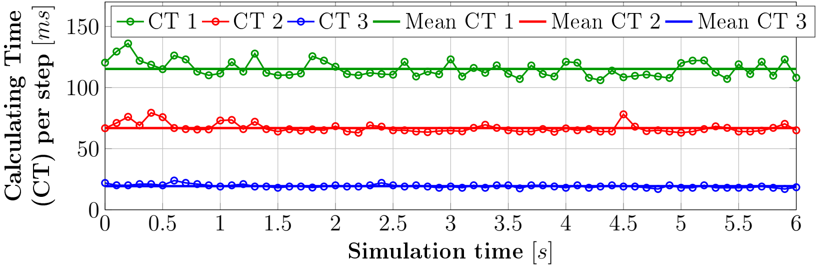
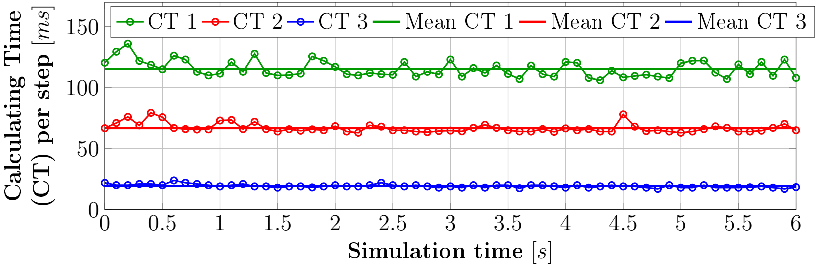


Calculating time in the 3 scenarios.

	Scen. 1	Scen. 2	Scen. 3
CT $[s]$	7.0285	4.0765	**1.1786**
CT per step $[ms]$	115.2216	66.8271	**19.3211**

Simulation results



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- An NMPC scheme for multicopters with semi-globally asymptotic stability
- The size of the terminal set is easily modified

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- To solve the trade-off between the size of the terminal set - the prediction horizon - the convergence time

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- Thanks to Prof. Fernando A. C. C. Fontes and Prof. Fernando Lobo Pereira for the discussions.

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THANK YOU

QUESTIONS AND DISCUSSIONS

Explicit solution of the Lyapunov equation

Sketch of the proof (cont.)

Now, we want: $M \succcurlyeq Q^* \Leftrightarrow (M - Q^*) \succcurlyeq \mathbf{0}$

$$\Leftrightarrow \mathbf{x}^\top (M - Q^*)\mathbf{x} = \sum_{q \in \{x,y,z\}} [(m_q - Q^*_{1_q})q^2 + (m_{v_q} - Q^*_{2_q})v_q^2 - 2Q^*_{3_q} q v_q] \geq 0$$

$$\Leftrightarrow (m_q - Q^*_{1_q})q^2 + (m_{v_q} - Q^*_{2_q})v_q^2 - 2Q^*_{3_q} q v_q \geq 0, \;\; \forall q \in \{x, y, z\}$$

$$\Leftrightarrow \begin{bmatrix} m_q - Q^*_{1_q} & -Q^*_{3_q} \\ -Q^*_{3_q} & m_{v_q} - Q^*_{2_q} \end{bmatrix} \succcurlyeq \mathbf{0}, \;\; \forall q \in \{x, y, z\}$$

Explicit solution of the Lyapunov equation

Positive semi-definiteness (Gantmacher 1960)

A quadratic form is positive semi-definite iff all the principal minors of its coefficient matrix are non-negative

Sketch of the proof (cont.)

$$\begin{bmatrix} m_q - Q^*_{1_q} & -Q^*_{3_q} \\ -Q^*_{3_q} & m_{v_q} - Q^*_{2_q} \end{bmatrix} \succcurlyeq \mathbf{0},\ \forall q \in \{x, y, z\}$$

$$\Leftrightarrow m_q - Q^*_{1_q} \geq 0, m_{v_q} - Q^*_{2_q} \geq 0, (m_q - Q^*_{1_q})(m_{v_q} - Q^*_{2_q}) \geq Q^{*2}_{3_q},\ \forall q \in \{x, y, z\}$$

From calculation: $\{Q^*_{1_q}, Q^*_{2_q} > 0 : q \in \{x, y, z\}\}$

We choose: $m_q \geq Q^*_{1_q} + |Q^*_{3_q}| > 0,\ m_{v_q} \geq Q^*_{2_q} + |Q^*_{3_q}| > 0$

$$\Rightarrow M \succcurlyeq Q^* \succ \mathbf{0}.$$

Explicit solution of the Lyapunov equation

Sketch of the proof (cont.)

$$\begin{bmatrix} \mathbf{0} & [K_p] \\ I_3 & [K_d] \end{bmatrix} \begin{bmatrix} [P_1] & [P_3] \\ [P_3] & [P_2] \end{bmatrix} + \begin{bmatrix} [P_1] & [P_3] \\ [P_3] & [P_2] \end{bmatrix} \begin{bmatrix} \mathbf{0} & I_3 \\ [K_p] & [K_d] \end{bmatrix}$$

$$+ \begin{bmatrix} [m] & \mathbf{0} \\ \mathbf{0} & [m_v] \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{cases} 2[K_p] \circ [P_3] + [m] & = \mathbf{0}, \\ [P_1] + [K_p] \circ [P_2] + [K_d] \circ [P_3] & = \mathbf{0}, \\ 2[P_3] + 2[K_d] \circ [P_2] + [m_v] & = \mathbf{0}. \end{cases}$$

$$\Rightarrow \begin{bmatrix} \mathbf{0} & \mathbf{0} & 2[K_p] \\ \mathbf{0} & 2[K_d] & 2\boldsymbol{I}_3 \\ \boldsymbol{I}_3 & [K_p] & [K_d] \end{bmatrix} \begin{bmatrix} \boldsymbol{P_1} \\ \boldsymbol{P_2} \\ \boldsymbol{P_3} \end{bmatrix} = - \begin{bmatrix} \boldsymbol{m} \\ \boldsymbol{m_v} \\ \mathbf{0} \end{bmatrix}$$

Element-wise product

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$A \circ B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \dots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \dots & a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{m1} & a_{m2}b_{m2} & \dots & a_{mn}b_{mn} \end{bmatrix}$$

Explicit solution of the Lyapunov equation

Sketch of the proof

$$\det(P - \lambda \boldsymbol{I}_6) = 0$$

$$\Rightarrow \lambda^2 \boldsymbol{I}_3 - \lambda([P_1] + [P_2]) + ([P_1] \circ [P_2] - [P_3] \circ [P_3]) = \mathbf{0}$$

$$\Rightarrow \left\{ \lambda^2 - \lambda(P_{1_q} + P_{2_q}) + (P_{1_q} P_{2_q} - P_{3_q}^2) = 0 : q \in \{x, y, z\} \right\}$$

$$\Rightarrow \{\lambda_{1_q}, \lambda_{2_q}\} = \left\{ \frac{1}{2} \left(P_{1_q} + P_{2_q} \pm \sqrt{(P_{1_q} - P_{2_q})^2 + 4P_{3_q}^2} \right) \right\}, \; q \in \{x, y, z\}$$

Explicit solution of the Lyapunov equation

Sketch of the proof (cont.)

$$\Big\{\lambda^2 - \lambda(P_{1_q} + P_{2_q}) + (P_{1_q}P_{2_q} - P^2_{3_q}) = 0 : q \in \{x, y, z\}\Big\}$$

$$\begin{cases} \lambda_{1_q} + \lambda_{2_q} = P_{1_q} + P_{2_q} = \dfrac{m_q(K^2_{d_q} - K_{p_q} + 1) + m_{v_q}(K^2_{p_q} - K_{p_q})}{2K_{p_q}K_{d_q}} > 0 \\ \lambda_{1_q}\lambda_{2_q} = P_{1_q}P_{2_q} - P^2_{3_q} = \dfrac{-m^2_q + m_q m_{v_q}(2K_{p_q} - K^2_{d_q}) - m^2_{v_q}K^2_{p_q}}{4K_{p_q}K^2_{d_q}} > 0 \end{cases}$$

$\Rightarrow \lambda_{1_q}$ and λ_{2_q} are positive $\Rightarrow P \succ \mathbf{0}$

Terminal region enlargement

$$\lambda_{\substack{min\\max}}(P) = \frac{1}{4}\left\{\left(\frac{k_d}{k_p} - \frac{1}{k_d} + \frac{1}{k_pk_d}\right)m_1 + \left(\frac{k_p-1}{k_d}\right)m_2 \mp \sqrt{\left[\left(\frac{k_d}{k_p} - \frac{1}{k_d} - \frac{1}{k_pk_d}\right)m_1 + \left(\frac{k_p+1}{k_d}\right)m_2\right]^2 + \frac{4}{k_p^2}m_1^2}\right\}$$

Define $\sigma \triangleq m_1/m_2 \Rightarrow \sigma > 0$

Let $\gamma_{\mp}(k_p, k_d, \sigma) \triangleq \left(k_d^2 - k_p + 1\right)\sigma + k_p^2 - k_p \mp \sqrt{\left[\left(k_d^2 - k_p - 1\right)\sigma + k_p^2 + k_p\right]^2 + 4k_d^2\sigma^2}$

$$\Rightarrow \bar{r}^2 = \frac{\lambda_{\mathsf{min}}(P)}{\lambda_{\mathsf{max}}(P)}r^2 = \frac{\gamma_-(k_p, k_d, \sigma)}{\gamma_+(k_p, k_d, \sigma)}r^2 = \frac{\gamma_-(k_p, k_d, \sigma)}{\gamma_+(k_p, k_d, \sigma)} \times \frac{U_{\mathsf{min}}^2}{k_p^2 + k_d^2}$$

with $U_{\mathsf{min}}^2 \triangleq \min_{q\in\{x,y,z\}}\left\{U_q^2\right\}$

Quasi-inifinite MPC (Chen and Allgöwer 1998) – Algorithm

1. **data** f, $\mathbf{x}_e$, $\mathbf{u}_e$
2. Calculate $A_{qs} = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}_e, \mathbf{u}_e) \in \mathbb{R}^{6\times 6}$, $B_{qs} = \frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}_e, \mathbf{u}_e) \in \mathbb{R}^{6\times 3}$
3. Choose the feedback gain $K_{qs} \in \mathbb{R}^{3\times 6}$
4. Calculate $A_{K_{qs}} = A_{qs} + B_{qs}K_{qs}$
5. Choose κ satisfying $0 < \kappa < -\lambda_{\max}(A_{K_{qs}})$ (Chen and Allgöwer 1998, eqn. (10))
6. Choose $Q \in \mathbb{S}^6_{++}$, $R \in \mathbb{S}^3_{+}$
7. Solve a Ricatti equation for P_{qs} (Chen and Allgöwer 1998, eqn. (9))
8. Find the largest $\alpha_1 > 0$ such that $K\mathbf{x} \in \mathbb{U}$ for $\mathbf{x}$ in $\mathbf{x}^\top P_{qs}\mathbf{x} \leq \alpha_1$
9. Construct the terminal invariant set $\mathcal{X}_{f_{qs}}$:

$$\mathcal{X}_{f_{qs}} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^\top P_{qs}\mathbf{x} \leq \alpha\},$$

with $\alpha \in (0, \alpha_1]$ is a solution of an optimization problem:

$$\Lambda = \max_{\mathbf{x}} \{\mathbf{x}^\top P_{qs}\left[f(\mathbf{x}, \mathbf{u}_{qs}) - A_{K_{qs}}\mathbf{x}\right] - \kappa \mathbf{x}^\top P_{qs}\mathbf{x}\},$$

$$\text{s.t.} \quad \mathbf{u}_{qs} \in \mathcal{U} \ \forall \mathbf{x} \in \mathcal{X}_{f_{qs}}, \ \mathbf{x}^\top P_{qs}\mathbf{x} \leq \alpha, \ \Lambda \leq 0.$$

10. **result** $\mathcal{X}_{f_{qs}}$

Simulation parameters

$$P_{qs} = \begin{bmatrix} 304.8837 & 0 & 0 & 147.3301 & 0 & 0 \\ 0 & 304.8837 & 0 & 0 & 147.3301 & 0 \\ 0 & 0 & 342.4938 & 0 & 0 & 166.1845 \\ 147.3301 & 0 & 0 & 145.0961 & 0 & 0 \\ 0 & 147.3301 & 0 & 0 & 145.0961 & 0 \\ 0 & 0 & 166.1845 & 0 & 0 & 164.9377 \end{bmatrix}$$